ORTHOGONAL POLYNOMIAL SOLUTIONS OF A CLASS OF FOURTH ORDER LINEAR DIFFERENTIAL EQUATIONS

By

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TO MY WIFE

For her patience,

interest and assistance

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INTRODUCTION

Several well-known polynomials have arisen as solutions to a class of linear differential equations of the second order. The polynomials of Jacobi, Hermite and Laguerre are classical examples that may originate in this manner. These polynomials become all the more interesting due to a number of properties which they have in common.

One of the more interesting of these properties is that of orthogonality, with respect to a weight function, of a set of polynomial solutions over a fundamental interval. The use of this property, in expanding an arbitrary function in a series of these polynomials, has proved to be a valuable contribution to the field of mathematics.

In this paper a class of linear differential equations of the fourth order will be considered. The first step will be the determination of conditions whereby the differential equation yields a set of polynomial solutions. The orthogonalization of this solution set and the subsequent formal expansion of an arbitrary function in a series of these orthogonal polynomial solutions are further objectives. Conditions for achieving these objectives will be derived. The separate cases for the finite, semi-infinite and infinite intervals are each discussed and the requirements for each case are determined.

Analogues of the classical orthogonal polynomials of Jacobi, Hermite and Laguerre will be obtained in this manner.

CHAPTER I

POLYNOMIAL SOLUTIONS

The Differential Equation

Consider the differential equation

$$(1.1) \qquad p(x)y_n^{iv} + q(x)y_n''' + r(x)y_n'' + s(x)y_n' + \lambda_n t(x)y_n = 0,$$

where the coefficients p, q, r, s and t, none of which vanish identically, are assumed to be real polynomials in the real variable x. Let λ_n be a polynomial in n.

Form of Solutions

Polynomials of the form

$$(1.2) \qquad y_n = \sum_{j=0}^{n} a_{jn} x^{n-j}, \quad a_{on} \neq 0, \quad n = 0, 1, 2, \ldots,$$

are desired as solutions of the differential equation (1.1). A solution set y_n, n = 0,1,2,..., can thus be written as

$$(1.3) \qquad \begin{cases} y_0 = a_{00} \\ y_1 = a_{01}x + a_{11} \\ y_2 = a_{02}x^2 + a_{12}x + a_{22} \\ \quad \cdots \\ y_n = a_{on}x^n + a_{1n}x^{n-1} + \cdots + a_{nn} \\ \quad \cdots \end{cases}$$

The forcing of the members of (1.3) as solutions of the differential equation (1.1) will impose certain restrictions on p, q, r, s and t of equation (1.1).

The solution $y_0 = a_{00}$. If $y_0 = a_{00}$ is to be a solution of equation (1.1), then

$$\lambda_0 t(x) a_{00} = 0. \tag{1.4}$$

Since $t(x) \not\equiv 0$ and $a_{00} \neq 0$, then $\lambda_0 = 0$. This requirement imposes no restrictions on p, q, r, s and t.

The solution $y_1 = a_{01}x + a_{11}$. If $y_1 = a_{01}x + a_{11}$ is to be a solution of equation (1.1), then

$$s(x) a_{01} + \lambda_1 t(x)(a_{01}x + a_{11}) \equiv 0. \tag{1.5}$$

Since $a_{01} \neq 0$, then $\lambda_1 \neq 0$, as s(x) would vanish identically for $\lambda_1 = 0$. Dividing equation (1.5) by t(x) yields

$$S(x) = \frac{s(x)}{t(x)} = -\frac{\lambda_1}{a_{01}}(a_{01}x + a_{11}) \not\equiv 0. \tag{1.6}$$

Hence, S(x) is a polynomial of degree one, and equation (1.1) becomes

$$p(x)y_n^{\text{IV}} + q(x)y_n''' + r(x)y_n'' + t(x)S(x)y_n' + \lambda_n t(x) y_n = 0. \tag{1.7}$$

The solution $y_2 = a_{02}x^2 + a_{12}x + a_{22}$. If $y_2 = a_{02}x^2 + a_{12}x + a_{22}$ is to be a solution of equation (1.7), then

$$\begin{aligned} 2r(x)a_{02} + t(x)S(x)(2a_{02}x + a_{12}) \\ + \lambda_2 t(x)(a_{02}x^2 + a_{12}x + a_{22}) = 0. \end{aligned} \tag{1.8}$$

Dividing equation (1.8) by t(x) yields

(1.9) $$R(x) = \frac{r(x)}{t(x)}$$

$$= -\frac{1}{2a_{02}}\left[S(x)(2a_{02}x + a_{12}) + \lambda_2(a_{02}x^2 + a_{12}x + a_{22})\right] \not\equiv 0.$$

Hence, R(x) is a polynomial of degree two, at most. Equation (1.7) now becomes

(1.10) $$p(x)y_n^{IV} + q(x)y_n''' + t(x)R(x)y_n'' + t(x)S(x)y_n' + \lambda_n t(x)y_n = 0.$$

The solution $y_3 = a_{03}x^3 + a_{13}x^2 + a_{23}x + a_{33}$. If this solution is to satisfy equation (1.10), then

(1.11) $$6q(x)a_{03} + t(x)R(x)(6a_{03}x + 2a_{13})$$

$$+ t(x)S(x)(3a_{03}x^2 + 2a_{13}x + a_{23})$$

$$+ \lambda_3 t(x)(a_{03}x^3 + a_{13}x^2 + a_{23}x + a_{33}) \equiv 0.$$

Dividing equation (1.11) by t(x) yields

(1.12) $$Q(x) = \frac{q(x)}{t(x)}$$

$$= -\frac{1}{6a_{03}}\left[R(x)(6a_{03}x + 2a_{13}) + S(x)(3a_{03}x^2 + 2a_{13}x + a_{23})\right.$$

$$\left. + \lambda_3(a_{03}x^3 + a_{13}x^2 + a_{23}x + a_{33})\right] \not\equiv 0.$$

Hence, Q(x) is a polynomial of degree three, at most, and equation (1.10) can now be written as

(1.13) $$p(x)y_n^{IV} + t(x)Q(x)y_n''' + t(x)R(x)y_n'' + t(x)S(x)y_n' + \lambda_n t(x)y_n = 0.$$

The solution $y_4 = a_{04}x^4 + a_{14}x^3 + a_{24}x^2 + a_{34}x + a_{44}$. If this solution is to satisfy equation (1.13), then

$$(1.14) \qquad 24\,p(x)a_{04} + t(x)Q(x)(24a_{04}x + 6a_{14})$$

$$+ t(x)R(x)(12a_{04}x^2 + 6a_{14}x + 2a_{24})$$

$$+ t(x)S(x)(4a_{04}x^3 + 3a_{14}x^2 + 2a_{24}x + a_{34})$$

$$+ \lambda_4 t(x)(a_{04}x^4 + a_{14}x^3 + a_{24}x^2 + a_{34}x + a_{44}) \equiv 0.$$

Dividing equation (1.14) by t(x) yields

$$(1.15) \qquad P(x) = \frac{p(x)}{t(x)}$$

$$= -\frac{1}{24a_{04}}\,[Q(x)(24a_{04}x + 6a_{14})$$

$$+ R(x)(12a_{04}x^2 + 6a_{14}x + 2a_{24})$$

$$+ S(x)(4a_{04}x^3 + 3a_{14}x^2 + 2a_{24}x + a_{34})$$

$$+ \lambda_4(a_{04}x^4 + a_{14}x^3 + a_{24}x^2 + a_{34}x + a_{44}) \not\equiv 0.$$

Hence, P(x) is a polynomial of degree four, at most, and equation (1.13) becomes

$$(1.16) \qquad t(x)P(x)y_n^{IV} + t(x)Q(x)y_n''' + t(x)R(x)y_n''$$

$$+ t(x)S(x)y_n' + \lambda_n t(x)y_n = 0.$$

Division by t(x) causes equation (1.16) to assume the form

$$(1.17) \qquad P(x)y_n^{IV} + G(x)y_n''' + R(x)y_n'' + S(x)y_n' + \lambda_n y_n = 0.$$

The solution $y_k = a_{0k}x^k + a_{1k}x^{k-1} + \cdots + a_{kk}$, $k = 5,6,7,\ldots$. In order for this solution to satisfy equation (1.17), then

$$
\begin{aligned}
(1.18) \qquad & P(x)[k(k-1)(k-2)(k-3)a_{0k}x^{k-4} + \cdots + 24a_{(k-4)k}] \\
& + Q(x)[k(k-1)(k-2)a_{0k}x^{k-3} + \cdots + 6a_{(k-3)k}] \\
& + R(x)[k(k-1)a_{0k}x^{k-2} + \cdots + 2a_{(k-2)k}] \\
& + S(x)[ka_{0k}x^{k-1} + \cdots + a_{(k-1)k}] \\
& + \lambda_k[a_{0k}x^k + \cdots + a_{kk}] \equiv 0.
\end{aligned}
$$

The left member of equation (1.18) is thus a polynomial of degree k, at most. By a proper choice of the coefficients a_{jk}, $j = 0,1,\ldots,k$, and λ_k, the left member can be made to vanish identically.

Summary

The differential equation (1.17) has polynomial solutions of the form of equation (1.2) if P(x) is a polynomial of degree four, at most; Q(x) is a polynomial of degree three, at most; R(x) is a polynomial of degree two, at most; S(x) is a polynomial of degree one, and $\lambda_0 = 0$.

CHAPTER II

ORTHOGONALITY OF THE SOLUTION SET

Derivation of Conditions

In the preceding chapter conditions on the polynomial coefficients P, Q, R and S of the differential equation (1.17) have been established such that a polynomial solution set of the form of equation (1.2) exists for the differential equation (1.17).

It is now desired to derive a set of conditions under which the solution set $\{y_n\}$, n = 0,1,2,..., will form an orthogonal system, with respect to a weight function $\rho(x)$, over the fundamental interval (α,β).

Formation of the Basic Equation

Consider the system of equations:

$$Py_n^{iv} + Qy_n''' + Ry_n'' + Sy_n' + \lambda_n y_n = 0, \tag{2.1}$$

$$Py_m^{iv} + Qy_m''' + Ry_m'' + Sy_m' + \lambda_m y_m = 0, \tag{2.2}$$

where $n \neq m$.

Multiplication of equations (2.1) and (2.2) by ρy_m and $-\rho y_n$, respectively, will yield the system:

$$\rho P y_n^{iv} y_m + \rho Q y_n''' y_m + \rho R y_n'' y_m + \rho S y_n' y_m + \lambda_n \rho y_n y_m = 0, \tag{2.3}$$

$$-\rho P y_m^{iv} y_n - \rho Q y_m''' y_n - \rho R y_m'' y_n - \rho S y_m' y_n - \lambda_m \rho y_n y_m = 0. \tag{2.4}$$

Addition of equations (2.3) and (2.4) gives

$$(2.5)\quad \rho P(y_n^{iv}y_m - y_m^{iv}y_n) + \rho Q(y_n'''y_m - y_m'''y_n) + \rho R(y_n''y_m - y_m''y_n)$$

$$+ \rho S(y_n'y_m - y_m'y_n) + (\lambda_n - \lambda_m)\partial y_n y_m = 0, \quad n \neq m.$$

Let the following change of variables be made,

$$(2.6)\qquad u = y_n'y_m - y_m'y_n,$$

so that:

$$(2.7)\qquad u' = y_n''y_m - y_m''y_n,$$

$$(2.8)\qquad u'' = y_n'''y_m + y_n''y_m' - y_m''y_n' - y_m'''y_n,$$

$$(2.9)\qquad u''' = y_n^{iv}y_m + 2y_n'''y_m' - 2y_m'''y_n' - y_m^{iv}y_n.$$

Also, let

$$(2.10)\qquad v = y_n''y_m' - y_m''y_n',$$

so that

$$(2.11)\qquad v' = y_n'''y_m' - y_m'''y_n'.$$

The substitution of equation (2.11) into equation (2.9) gives

$$(2.12)\qquad y_n^{iv}y_m - y_m^{iv}y_n = u''' - 2v'.$$

Substitution of equation (2.10) into (2.8) gives

$$(2.13)\qquad y_n'''y_m - y_m'''y_n = u'' - v.$$

Equations (2.13), (2.12), (2.7) and (2.6) substituted into equation (2.5) give

(2.14) $\rho P(u''' - 2v') + \rho Q(u'' - v) + \rho R u' + \rho S u + (\lambda_n - \lambda_m)\rho y_n y_m = 0, \; n \neq m,$

or more simply,

(2.15) $\rho P u''' + \rho Q u'' + \rho R u' + \rho S u - 2\rho P v' - \rho Q v + (\lambda_n - \lambda_m)\rho y_n y_m = 0, \; n \neq m.$

Equation (2.15) is to be known as the basic equation for orthogonality.

Derivative Form of the Basic Equation

Consider the following relationships:

(2.16) $(\rho P u'')' = (\rho P)' u'' + \rho P u''',$

(2.17) $\{[\rho Q - (\rho P)']u'\}' = [(\rho Q)' - (\rho P)'']u' + [\rho Q - (\rho P)']u'',$

(2.18) $\{[\rho R - (\rho Q)' + (\rho P)'']u\}' = [(\rho R)' - (\rho Q)'' + (\rho P)''']u + [(\rho R) - (\rho Q)' + (\rho P)'']u'.$

Choose the condition

(2.19) $$2(\rho P)' \equiv \rho Q.$$

The substitution of equations (2.16), (2.17), (2.18) and (2.19) into the basic equation (2.15) will yield

(2.20) $$(\rho P u'')' - (\rho P)' u'' + \{[\rho Q - (\rho P)']u'\}' - (\rho Q)' u'$$
$$+ (\rho P)'' u' + (\rho P)' u'' + \{[\rho R - (\rho Q)' + (\rho P)'']u\}' - (\rho R)' u$$
$$+ (\rho Q)'' u - (\rho P)''' u + (\rho Q)' u' - (\rho P)'' u' + \rho S u - 2\rho P v'$$
$$- 2(\rho P)' v + (\lambda_n - \lambda_m)\rho y_n y_m = 0, \; n \neq m.$$

Equation (2.20) can be simplified to

$$(2.21) \qquad (\rho P u'')' + \{[\rho Q - (\rho P)']u'\}' + \{[\rho R - (\rho Q)' + (\rho P)'']u\}'$$

$$+ [\rho S - (\rho R)' + (\rho Q)'' - (\rho P)''']u - 2(\rho P \sigma)'$$

$$+ (\lambda_n - \lambda_m)\rho y_n y_m = 0, \quad n \neq m.$$

Equation (2.21) is the derivative form of the basic equation.

Integration of the Derivative Form of the Basic Equation

Let equation (2.21) be integrated, with respect to x, from $x = \alpha$ to $x = \beta$. That is, let

$$(2.22) \qquad \int_\alpha^\beta (\rho P u'')' dx + \int_\alpha^\beta \{[\rho Q - (\rho P)']u'\}' dx$$

$$+ \int_\alpha^\beta \{[\rho R - (\rho Q)' + (\rho P)'']u\}' dx + \int_\alpha^\beta [\rho S - (\rho R)' + (\rho Q)'' - (\rho P)''']u\, dx$$

$$-2\int_\alpha^\beta (\rho P \sigma)' dx + (\lambda_n - \lambda_m)\int_\alpha^\beta \rho y_n y_m dx = 0, \quad n \neq m.$$

Performing the indicated integration in equation (2.22) gives

$$(2.23) \qquad (\rho P u'')\Big|_\alpha^\beta + \{[\rho Q - (\rho P)']u'\}\Big|_\alpha^\beta$$

$$+ \{[\rho R - (\rho Q)' + (\rho P)'']u\}\Big|_\alpha^\beta + \int_\alpha^\beta [\rho S - (\rho R)' + (\rho Q)'' - (\rho P)''']u\, dx$$

$$-2(\rho P \sigma)\Big|_\alpha^\beta + (\lambda_n - \lambda_m)\int_\alpha^\beta \rho y_n y_m dx = 0, \quad n \neq m.$$

The Placing of Conditions on the Integrated Basic Equation

It is desired to obtain the result

$$\int_{\alpha}^{\beta} \rho y_n y_m \, dx = 0, \quad n \neq m,$$

which is the usual definition of an orthogonal system. Consequently, choose the conditions:

(2.24) $\rho P = 0$ at $x = \alpha$ and $x = \beta$,

(2.25) $\rho Q - (\rho P)' = 0$ at $x = \alpha$ and $x = \beta$,

(2.26) $\rho R - (\rho Q)' + (\rho P)'' = 0$ at $x = \alpha$ and $x = \beta$,

(2.27) $\rho S - (\rho R)' + (\rho Q)'' - (\rho P)''' \equiv 0.$

These conditions, together with the previously imposed condition of equation (2.19) which states

$$2(\rho P)' \equiv \rho Q,$$

are imposed on equation (2.23). The result is

(2.28) $$(\lambda_n - \lambda_m) \int_{\alpha}^{\beta} \rho y_n y_m \, dx = 0, \quad n \neq m.$$

Furthermore, if the parameter values λ_n are all distinct,

(2.29) $$\int_{\alpha}^{\beta} \rho y_n y_m \, dx = 0, \quad n \neq m.$$

Summary

By considering equation (2.19) and its derivatives, it is possible to simplify equations (2.24), (2.25), (2.26) and (2.27), and obtain:

(i) $\rho P = 0$ at α and β,

(ii) $(\rho P)' = 0$ at α and β,

(iii) $\rho R - (\rho P)'' = 0$ at α and β,

(iv) $\rho S \equiv (\rho R)' - (\rho P)'''$,

(v) $\rho Q \equiv 2(\rho P)'$.

These five conditions will henceforth be referred to as orthogonality conditions (i), (ii), (iii), (iv) and (v).

It has now been established that under these five orthogonality conditions the solution set $\{y_n\}$, n = 0,1,2,..., of the differential equation (1.17) will form an orthogonal system, with respect to the weight function $\rho(x)$, over the fundamental interval (α,β).

This may be more explicitly expressed by stating that if $\rho(x)$ is assumed to be non-negative in the interval (α,β), then the set $\{\rho^{\frac{1}{2}}y_n\}$, n = 0,1,2,..., forms an orthogonal system over the fundamental interval (α,β).

CHAPTER III

EXPANSION OF AN ARBITRARY FUNCTION IN SERIES

Assume that an arbitrary function f(x) can be expanded as

(3.1) $$f(x) = a_0 y_0 + a_1 y_1 + \cdots + a_n y_n + \cdots + a_m y_m + \cdots,$$

where the a_k are constants to be determined and the y_k are solutions of equation (1.17). The determination of the constants a_k, $k = 0,1,2,\ldots$, will enable f(x) to be formally expanded as a series in the polynomial solutions of differential equation (1.17).

Multiply both sides of equation (3.1) by ρy_n to obtain

(3.2) $$\rho f(x) y_n = a_0 \rho y_0 y_n + \cdots + a_n \rho y_n^2 + \cdots + a_m \rho y_n y_m + \cdots.$$

Integration of equation (3.2), with respect to x, over the fundamental interval of orthogonality (α, β) will yield

(3.3) $$\int_\alpha^\beta \rho f(x) y_n dx = a_0 \int_\alpha^\beta \rho y_0 y_n dx + a_1 \int_\alpha^\beta \rho y_1 y_n dx + \cdots$$
$$+ a_n \int_\alpha^\beta \rho y_n^2 dx + \cdots$$
$$+ a_m \int_\alpha^\beta \rho y_n y_m dx + \cdots.$$

Equation (2.29) may now be utilized so that every term of the right member of equation (3.3), with the exception of the integrated square, vanishes. Thus,

$$\int_{\alpha}^{\beta} \rho f(x) y_n \, dx = a_n \int_{\alpha}^{\beta} \rho y_n^2 \, dx. \tag{3.4}$$

Each a_n, $n = 0,1,2,\ldots$, can now be uniquely determined, subject only to the integrability of the expressions in equation (3.4).

Hence, an arbitrary function of x can be formally expanded as a series in the solution set $\{y_n\}$, $n = 0,1,2,\ldots$, of the differential equation (1.17).

CHAPTER IV

SOLUTIONS IN THE FINITE INTERVAL

The Finite Interval

The fundamental interval (α, β) may conceivably extend in either direction, or both directions, to infinity. These situations will be discussed in Chapters V and VI. This chapter, however, will be devoted to the consideration of the finite interval (α, β), where $\alpha < \beta$.

The Weight Function $\rho(x)$

The satisfaction of the five orthogonality conditions, developed in Chapter II and stated in its summary, depends on the choice of $\rho(x)$. Conversely, the choice of $\rho(x)$ is vital only with respect to the part it plays in the satisfaction of these orthogonality conditions.

A close scrutiny of these orthogonality conditions indicates a form of $\rho(x)$ which is sufficient to accomplish this aim.

From orthogonality condition (i) it is seen that the possibility of $\rho(x)$ vanishing at $x = \alpha$ and at $x = \beta$ must be included.

Since orthogonality condition (ii) states that $(\rho P)' = \rho' P + \rho P' = 0$ at α and β, then the possibility that $\rho'(x)$ vanishes at $x = \alpha$ and at $x = \beta$ must not be excluded.

From orthogonality condition (iii), which states that $\rho R - (\rho P)'' = \rho R - \rho'' P - 2\rho' P' - \rho P'' = 0$ at α and β, it is seen that ρ must be such that $\rho''(x)$ might vanish at $x = \alpha$ and $x = \beta$.

Orthogonality conditions (iv) and (v) are identities which state, respectively, that $\rho S \equiv \rho' R + \rho R' - \rho''' P - 3\rho'' P' - 3\rho' P'' - \rho P'''$ and $\rho Q \equiv 2\rho' P + 2\rho P'$. The polynomial nature of Q and S indicates that ρ must be a factor of the right members of both of these identities. This is feasible if ρ', ρ'' and ρ''' can each be expressed as the product of ρ and a rational function, where the denominator of the rational function divides the numerator of the term in which it appears.

Hence, consider a choice of

$$\rho = (x-\alpha)^g (x-\beta)^h, \tag{4.1}$$

where g and h are real numbers. Successive differentiations of equation (4.1) yield:

$$\rho' = g(x-\alpha)^{g-1}(x-\beta)^h + h(x-\alpha)^g(x-\beta)^{h-1} = (x-\alpha)^g(x-\beta)^h\left[\frac{g(x-\beta)+h(x-\alpha)}{(x-\alpha)(x-\beta)}\right], \tag{4.2}$$

$$\begin{aligned} \rho'' &= g(g-1)(x-\alpha)^{g-2}(x-\beta)^h + 2gh(x-\alpha)^{g-1}(x-\beta)^{h-1} + h(h-1)(x-\alpha)^g(x-\beta)^{h-2} \\ &= (x-\alpha)^g(x-\beta)^h\left[\frac{g(g-1)(x-\beta)^2 + 2gh(x-\alpha)(x-\beta) + h(h-1)(x-\alpha)^2}{(x-\alpha)^2(x-\beta)^2}\right], \end{aligned} \tag{4.3}$$

$$\begin{aligned} \rho''' &= g(g-1)(g-2)(x-\alpha)^{g-3}(x-\beta)^h + 3gh(g-1)(x-\alpha)^{g-2}(x-\beta)^{h-1} \\ &\quad + 3gh(h-1)(x-\alpha)^{g-1}(x-\beta)^{h-2} + h(h-1)(h-2)(x-\alpha)^g(x-\beta)^{h-3} \\ &= (x-\alpha)^g(x-\beta)^h\left[\frac{g(g-1)(g-2)(x-\beta)^3 + 3gh(g-1)(x-\alpha)(x-\beta)^2}{(x-\alpha)^3(x-\beta)^3}\right. \\ &\quad \left. + \frac{3gh(h-1)(x-\alpha)^2(x-\beta) + h(h-1)(h-2)(x-\alpha)^3}{(x-\alpha)^3(x-\beta)^3}\right]. \end{aligned} \tag{4.4}$$

Consequently, the choice of $\rho = (x-\alpha)^g(x-\beta)^h$, with the proper restrictions on g and h, will serve to satisfy the orthogonality conditions.

The Leading Coefficient $P(x)$

The determination of g and h in equation (4.1) is contingent on whether or not $P(x) = 0$ has α and/or β as roots. A number of possibilities exists with regard to the roots of $P(x) = 0$. Investigation of each of these possibilities will determine the form of the coefficients of the differential equation (1.17).

The assumption that $P(\alpha) \neq 0$. If $P(x) = 0$ does not have α as a root, then application of orthogonality condition (i) yields

$$\rho P = (x-\alpha)^g(x-\beta)^h P = 0 \tag{4.5}$$

at $x = \alpha$. Since $P(\alpha) \neq 0$, by assumption, and since $\alpha < \beta$, then

$$g > 0. \tag{4.6}$$

Application of orthogonality condition (v) gives

$$(x-\alpha)^g(x-\beta)^h Q \equiv \lambda\left[(x-\alpha)^g(x-\beta)^h P\right]' \tag{4.7}$$

$$\equiv \lambda\left[g(x-\alpha)^{g-1}(x-\beta)^h P + h(x-\alpha)^g(x-\beta)^{h-1}P + (x-\alpha)^g(x-\beta)^h P'\right],$$

or

$$Q \equiv \lambda(x-\alpha)^{-1}(x-\beta)^{-1}\left[g(x-\beta)P + h(x-\alpha)P + (x-\alpha)(x-\beta)P'\right]. \tag{4.8}$$

The polynomial nature of Q requires that $(x-\alpha)$ divide the expression in brackets in equation (4.8), but if this is true, then

$$g(\alpha-\beta)P(\alpha) = 0. \tag{4.9}$$

From the inequality (4.6), $g > 0$. By assumption, $P(\alpha) \neq 0$ and . Equation (4.9) is thus a contradiction. Hence, the assumption that $P(\alpha) \neq 0$ is not valid.

The assumption that $P(\beta) \neq 0$. This assumption leads to a contradiction in a manner similar to that of the preceding section. This can be seen by interchanging the roles of α and β, and of g and h in the proof of contradiction of the assumption in the preceding section.

Consequently, $P(x) = 0$ must have both α and β as simple roots, at least.

The assumption that $P(x) = 0$ has α as a simple root. From the preceding section it is seen that $P(x) = (x-\alpha)(x-\beta)U(x)$ and, by the assumption in this section, $U(\alpha) \neq 0$.

Application of orthogonality condition (i) thus yields

$$\rho P = (x-\alpha)^{g+1}(x-\beta)^{h+1}U = 0 \tag{4.10}$$

at $x = \alpha$. Since $U(\alpha) \neq 0$, by assumption, and since $\alpha < \beta$, then

$$g > -1. \tag{4.11}$$

The application of orthogonality condition (ii) will give

$$\begin{aligned}(\rho P)' &= (g+1)(x-\alpha)^{g}(x-\beta)^{h+1}U + (h+1)(x-\alpha)^{g+1}(x-\beta)^{h}U \\ &\quad + (x-\alpha)^{g+1}(x-\beta)^{h+1}U' \\ &= (x-\alpha)^{g}(x-\beta)^{h}\left[(g+1)(x-\beta)U + (h+1)(x-\alpha)U \right. \\ &\quad \left. + (x-\alpha)(x-\beta)U'\right] = 0\end{aligned} \tag{4.12}$$

at $x = \alpha$. If $g > 0$, then $(\rho P)' = 0$ at $x = \alpha$. However, if $-1 < g \leqq 0$, then $(g+1)(\alpha-\beta)U(\alpha) = 0$. By inequality (4.11), $g > -1$. By assumption, $U(\alpha) \neq 0$ and $\alpha < \beta$. The equation $(g+1)(\alpha-\beta)U(\alpha) = 0$ is thus a contradiction. Hence,

$$g > 0. \tag{4.13}$$

Orthogonality condition (iii) states

$$(4.14) \quad \rho R - (\rho P)'' = (x-\alpha)^g (x-\beta)^h R - [(x-\alpha)^{g+1}(x-\beta)^{h+1} U]''$$

$$= (x-\alpha)^g (x-\beta)^h R - [g(g+1)(x-\alpha)^{g-1}(x-\beta)^{h+1} U$$

$$+ 2(g+1)(h+1)(x-\alpha)^g (x-\beta)^h U + 2(g+1)(x-\alpha)^g (x-\beta)^{h+1} U'$$

$$+ h(h+1)(x-\alpha)^{g+1}(x-\beta)^{h-1} U + 2(h+1)(x-\alpha)^{g+1}(x-\beta)^h U'$$

$$+ (x-\alpha)^{g+1}(x-\beta)^{h+1} U'']$$

$$= (x-\alpha)^{g-1}(x-\beta)^{h-1}[(x-\alpha)(x-\beta)R - g(g+1)(x-\beta)^2 U$$

$$- 2(g+1)(h+1)(x-\alpha)(x-\beta)U - 2(g+1)(x-\alpha)(x-\beta)^2 U'$$

$$- h(h+1)(x-\alpha)^2 U - 2(h+1)(x-\alpha)^2(x-\beta)U'$$

$$- (x-\alpha)^2(x-\beta)^2 U''] = 0$$

at $x = \alpha$. If $g > 1$, then $\rho R - (\rho P)'' = 0$ at $x = \alpha$. However, if $0 < g \leq 1$, then $-g(g+1)(\alpha-\beta)^2 U(\alpha) = 0$. By inequality (4.13), $g > 0$. By assumption, $U(\alpha) \neq 0$ and $\alpha < \beta$. The equation $-g(g+1)(\alpha-\beta)^2 U(\alpha) = 0$ is thus a contradiction. Hence,

$$(4.15) \qquad g > 1.$$

It is now seen that each of orthogonality conditions (i), (ii) and (iii) are satisfied if equation (4.15) holds. The application of orthogonality condition (iv), however, will result in a contradiction of the assumption of this section. This contradiction is established in the work that follows.

The application of orthogonality condition (iv) will give

$$(4.16)\quad (x-\alpha)^g(x-\beta)^h S \equiv [(x-\alpha)^g(x-\beta)^h R]' - [(x-\alpha)^{g+1}(x-\beta)^{h+1}U]'''$$

$$\equiv [g(x-\alpha)^{g-1}(x-\beta)^h R + h(x-\alpha)^g(x-\beta)^{h-1}R + (x-\alpha)^g(x-\beta)^h R']$$

$$-[g(g+1)(g-1)(x-\alpha)^{g-2}(x-\beta)^{h+1}U + 3g(g+1)(h+1)(x-\alpha)^{g-1}(x-\beta)^h U$$

$$+3g(g+1)(x-\alpha)^{g-1}(x-\beta)^{h+1}U' + 3h(h+1)(g+1)(x-\alpha)^g(x-\beta)^{h-1}U$$

$$+6(g+1)(h+1)(x-\alpha)^g(x-\beta)^h U' + 3(g+1)(x-\alpha)^g(x-\beta)^{h+1}U''$$

$$+h(h+1)(h-1)(x-\alpha)^{g+1}(x-\beta)^{h-2}U + 3h(h+1)(x-\alpha)^{g+1}(x-\beta)^{h-1}U'$$

$$+3(h+1)(x-\alpha)^{g+1}(x-\beta)^h U'' + (x-\alpha)^{g+1}(x-\beta)^{h+1}U'''],$$

or

$$(4.17)\quad S \equiv (x-\alpha)^{-2}(x-\beta)^{-2}[g(x-\alpha)(x-\beta)^2 R + h(x-\alpha)^2(x-\beta)R + (x-\alpha)^2(x-\beta)^2 R'$$

$$-g(g+1)(g-1)(x-\beta)^3 U - 3g(g+1)(h+1)(x-\alpha)(x-\beta)^2 U$$

$$-3g(g+1)(x-\alpha)(x-\beta)^3 U' - 3h(h+1)(g+1)(x-\alpha)^2(x-\beta)U$$

$$-6(g+1)(h+1)(x-\alpha)^2(x-\beta)^2 U' - 3(g+1)(x-\alpha)^2(x-\beta)^3 U''$$

$$-h(h+1)(h-1)(x-\alpha)^3 U - 3h(h+1)(x-\alpha)^3(x-\beta)U'$$

$$-3(h+1)(x-\alpha)^3(x-\beta)^2 U'' - (x-\alpha)^3(x-\beta)^3 U'''].$$

The polynomial nature of S requires that $(x-\alpha)$ divide the expression in brackets in equation (4.17), but if this is true, then

$$(4.18)\quad -g(g+1)(g-1)(\alpha-\beta)^3 U(\alpha) = 0.$$

From inequality (4.15), $g > 1$. By assumption, $U(\alpha) \neq 0$ and $\alpha < \beta$. Equation (4.18) is thus a contradiction. Hence, the assumption that $P(x) = 0$ has α as a simple root is not valid.

The assumption that $P(x) = 0$ has β as a simple root. This assumption is not valid either, as can be seen by interchanging the roles of α and β, and of g and h in the proof of contradiction of the assumption in the preceding section.

Verification that $P = A(x-\alpha)^2(x-\beta)^2$, where A is a constant. The contradictions of the assumptions made in the four preceding sections leave only one possibility; $P(x) = 0$ has both α and β as double roots. This is seen to be the case since in Chapter I the coefficient P was determined as a polynomial of degree four, at most.

Let the five orthogonality conditions be applied when ρ is chosen as $\rho = (x-\alpha)^g(x-\beta)^h$ and $P = A(x-\alpha)^2(x-\beta)^2$.

Orthogonality condition (i) requires

(4.19) $$\rho P = A(x-\alpha)^{g+2}(x-\beta)^{h+2} = 0$$

at $x = \alpha$ and at $x = \beta$. Now $A \neq 0$, since $P(x) \equiv 0$ if $A = 0$. Also, since $\alpha < \beta$ it is seen that

(4.20) $$g > -2$$

and

(4.21) $$h > -2.$$

Application of orthogonality condition (ii) yields

(4.22) $$(\rho P)' = A\left[(g+2)(x-\alpha)^{g+1}(x-\beta)^{h+2} + (h+2)(x-\alpha)^{g+2}(x-\beta)^{h+1}\right]$$

$$= A(x-\alpha)^{g+1}(x-\beta)^{h+1}\left[(g+2)(x-\beta) + (h+2)(x-\alpha)\right] = 0$$

at $x = \alpha$ and at $x = \beta$. If $g > -1$, then $(\rho P)' = 0$ at $x = \alpha$. If $-2 < g \leq -1$, then $(g+2)(\alpha-\beta) = 0$. Since $\alpha < \beta$ and, by inequality (4.20), $g > -2$, then $(g+2)(\alpha-\beta) \neq 0$. Hence,

(4.23) $$g > -1.$$

Also, if $h > -1$, then $(\rho P)' = 0$ at $x = \beta$. If $-2 < h \leq -1$, then $(h+2)(\beta-\alpha) = 0$. Since $\alpha < \beta$ and, by inequality (4.21), $h > -2$, then $(h+2)(\beta-\alpha) \neq 0$. Hence,

(4.24) $$h > -1.$$

By applying orthogonality condition (v), it is seen that

(4.25) $$(x-\alpha)^g (x-\beta)^h Q \equiv 2A(x-\alpha)^{g+1}(x-\beta)^{h+1}\left[(g+2)(x-\beta) + (h+2)(x-\alpha)\right],$$

or

(4.26) $$Q \equiv 2A(x-\alpha)(x-\beta)\left[(g+2)(x-\beta) + (h+2)(x-\alpha)\right].$$

When orthogonality condition (iii) is applied, then

(4.27) $$\rho R - (\rho P)'' = (x-\alpha)^g (x-\beta)^h R - A\left[(g+1)(g+2)(x-\alpha)^g (x-\beta)^{h+2}\right.$$

$$\left. + 2(g+2)(h+2)(x-\alpha)^{g+1}(x-\beta)^{h+1} + (h+1)(h+2)(x-\alpha)^{g+2}(x-\beta)^h\right]$$

$$= (x-\alpha)^g (x-\beta)^h \left[R - A(g+1)(g+2)(x-\beta)^2 - 2A(g+2)(h+2)(x-\alpha)(x-\beta)\right.$$

$$\left. - A(h+1)(h+2)(x-\alpha)^2\right] = 0$$

at $x = \alpha$ and at $x = \beta$. If $g > 0$, then $\rho R - (\rho P)'' = 0$ at $x = \alpha$. If $-1 < g \leqq 0$, then

$$R(\alpha) - A(g+1)(g+2)(\alpha-\beta)^2 = 0. \tag{4.28}$$

If $h > 0$, then $\rho R - (\rho P)'' = 0$ at $x = \beta$. If $-1 < h \leqq 0$, then

$$R(\beta) - A(h+1)(h+2)(\beta-\alpha)^2 = 0. \tag{4.29}$$

Only orthogonality condition (iv) now remains to be satisfied. If this condition is applied, then

$$\begin{aligned}(x-\alpha)^g(x-\beta)^h S &\equiv [(x-\alpha)^g(x-\beta)^h R]' - [A(x-\alpha)^{g+2}(x-\beta)^{h+2}]''' \\ &\equiv g(x-\alpha)^{g-1}(x-\beta)^h R + h(x-\alpha)^g(x-\beta)^{h-1}R \\ &\quad +(x-\alpha)^g(x-\beta)^h R' - A[g(g+1)(g+2)(x-\alpha)^{g-1}(x-\beta)^{h+2} \\ &\quad +3(g+1)(g+2)(h+2)(x-\alpha)^g(x-\beta)^{h+1} \\ &\quad +3(h+1)(h+2)(g+2)(x-\alpha)^{g+1}(x-\beta)^h \\ &\quad +h(h+1)(h+2)(x-\alpha)^{g+2}(x-\beta)^{h-1}],\end{aligned} \tag{4.30}$$

or

$$\begin{aligned}S \equiv (x-\alpha)^{-1}(x-\beta)^{-1}\,[&g(x-\beta)R + h(x-\alpha)R + (x-\alpha)(x-\beta)R' \\ &-Ag(g+1)(g+2)(x-\beta)^3 - 3A(g+1)(g+2)(h+2)(x-\alpha)(x-\beta)^2 \\ &-3A(h+1)(h+2)(g+2)(x-\alpha)^2(x-\beta) - Ah(h+1)(h+2)(x-\alpha)^3].\end{aligned} \tag{4.31}$$

The polynomial nature of S will require that $(x-\alpha)(x-\beta)$ divide the expression in brackets in equation (4.31). If $(x-\alpha)$ divides this expression, then

(4.32) $$g(\alpha-\beta)R(\alpha) - Ag(g+1)(g+2)(\alpha-\beta)^3 = 0,$$

or

(4.33) $$g(\alpha-\beta)[R(\alpha) - A(g+1)(g+2)(\alpha-\beta)^2] = 0.$$

So, since $\alpha < \beta$, either $g = 0$ or $R(\alpha) - A(g+1)(g+2)(\alpha-\beta)^2 = 0$. By consideration of equation (4.28) it is seen that if $g = 0$, then $R(\alpha) - A(g+1)(g+2)(\alpha-\beta)^2 = 0$. Therefore,

(4.34) $$R(\alpha) = A(g+1)(g+2)(\alpha-\beta)^2, \quad g > -1.$$

If $(x-\beta)$ divides the expression in brackets in equation (4.31), then

(4.35) $$h(\beta-\alpha)R(\beta) - Ah(h+1)(h+2)(\beta-\alpha)^3 = 0,$$

or

(4.36) $$h(\beta-\alpha)[R(\beta) - A(h+1)(h+2)(\beta-\alpha)^2] = 0.$$

Therefore, since $\alpha < \beta$, either $h = 0$ or $R(\beta) - A(h+1)(h+2)(\beta-\alpha)^2 = 0$. By consideration of equation (4.29) it is seen that if $h = 0$, then $R(\beta) - A(h+1)(h+2)(\beta-\alpha)^2 = 0$. Therefore,

(4.37) $$R(\beta) = A(h+1)(h+2)(\beta-\alpha)^2, \quad h > -1.$$

Equation (4.31) can now be written as

$$(4.38)\quad S \equiv \frac{gR - Ag(g+1)(g+2)(x-\beta)^2}{x-\alpha} + \frac{hR - Ah(h+1)(h+2)(x-\alpha)^2}{x-\beta}$$

$$+R' - 3A(g+1)(g+2)(h+2)(x-\beta) - 3A(h+1)(h+2)(g+2)(x-\alpha)$$

and, under the conditions imposed in equations (4.34) and (4.37), S is a first degree polynomial.

Summary

The choice of $\rho(x) = (x-\alpha)^g(x-\beta)^h$, where $g > -1$ and $h > -1$, will serve as a weight function in the orthogonalization of the solution set $\{y_n\}$, $n = 0,1,2,\ldots,$ of the differential equation (1.17) over the fundamental interval (α,β) where both α and β are finite.

However, this choice of $\rho(x)$ will require a number of restrictions on the coefficients P, Q, R and S of the differential equation (1.17). With regard to P it has already been determined that

$$P = A(x-\alpha)^2(x-\beta)^2.$$

Consequently, with this determination of P, equation (4.26) gives

$$Q = 2A(x-\alpha)(x-\beta)[(g+2)(x-\beta) + (h+2)(x-\alpha)].$$

The coefficient R is of the form

$$R = B_1x^2 + B_2x + B_3,$$

where, from equations (4.34) and (4.37) respectively,

$$R(\alpha) = A(g+1)(g+2)(\alpha-\beta)^2,$$

$$R(\beta) = A(h+1)(h+2)(\beta-\alpha)^2.$$

Equation (4.38) gives

$$S \equiv \frac{gR - Ag(g+1)(g+2)(x-\beta)^2}{x-\alpha} + \frac{hR - Ah(h+1)(h+2)(x-\alpha)^2}{x-\beta}$$

$$+ R' - 3A(g+1)(g+2)(h+2)(x-\beta) - 3A(h+1)(h+2)(g+2)(x-\alpha).$$

Examples of the Finite Interval

Example 1

Let $g = h = 0$, $\alpha = -1$ and $\beta = 1$. Then

$$\rho = 1.$$

$$P = A(x+1)^2(x-1)^2 = A(x^4 - 2x^2 + 1).$$

$$Q = 2A(x+1)(x-1)(4x) = 8A(x^3 - x).$$

$$R = B_1 x^2 + B_2 x + B_3,$$

where

$$R(-1) = 8A = B_1 - B_2 + B_3,$$

$$R(1) = 8A = B_1 + B_2 + B_3,$$

so that $B_2 = 0$ and $B_3 = 8A - B_1$. Thus,

$$R = B_1 x^2 + (8A - B_1).$$

$$S = (2B_1 - 24A)x.$$

The differential equation is

$$A(x^4 - 2x^2 + 1)y_n^{iv} + A(8x^3 - 8x)y_n''' + [B_1 x^2 + (8A - B_1)]y_n''$$

$$+ (2B_1 - 24A)x y_n' + \lambda_n y_n = 0.$$

Solutions of the form

$$y_n = \sum_{j=0}^{n} a_{jn} x^{n-j}, \quad a_{0n} \neq 0,$$

will be obtained.

Consequently,

$$
\begin{array}{r|l}
\lambda_n & y_n = a_{0n}x^n + \cdots + a_{nn}\ , \quad a_{0n} \neq 0 \\
(2B_1 - 24A)\,x & y_n' = n a_{0n} x^{n-1} + \cdots + a_{(n-1)n} \\
B_1 x^2 + (8A - B_1) & y_n'' = n(n-1) a_{0n} x^{n-2} + \cdots + 2a_{(n-2)n} \\
A(8x^3 - 8x) & y_n''' = n(n-1)(n-2) a_{0n} x^{n-3} + \cdots + 6a_{(n-3)n} \\
A(x^4 - 2x^2 + 1) & y_n^{iv} = n(n-1)(n-2)(n-3) a_{0n} x^{n-4} + \cdots + 24a_{(n-4)n}. \\
\hline
\end{array}
$$

The highest power of x that appears is n. The coefficient of x^n is

$$\left[\lambda_n + (2B_1 - 24A)n + B_1 n(n-1) + 8A\,n(n-1)(n-2) + A n(n-1)(n-2)(n-3)\right] a_{0n} = 0.$$

Since $a_{0n} \neq 0$, then

$$\lambda_n = -n\left[2B_1 - 24A + B_1(n-1) + 8A(n-1)(n-2) + A(n-1)(n-2)(n-3)\right]$$

$$= -n\left[An^3 + 2An^2 + (B_1 - 13A)n + B_1 - 14A\right].$$

The choices of A and B_1 are arbitrary, with the exception that $B_1 \neq 12A$ since $S \equiv 0$ if $B_1 = 12A$. Hence, choose $A = 1$ and $B_1 = 14$ and thus simplify λ_n to

$$\lambda_n = -n(n^3 + 2n^2 + n) = -n^2(n+1)^2.$$

The differential equation can now be written as

$$(x^4 - 2x^2 + 1)y_n^{iv} + (8x^3 - 8x)y_n''' + (14x^2 - 6)y_n'' + 4xy_n' - n^2(n+1)^2 y_n = 0.$$

For each choice of n, where n = 0,1,2,..., a polynomial solution of degree n will be obtained. The determinations of a few of these solutions follow.

Let n = 0. Then

$$y_0 = a_{00},$$

$$y_0' = y_0'' = y_0''' = y_0^{iv} = 0,$$

so that

$$0 \cdot a_{00} = 0.$$

Thus, a_{00} is arbitrary and

$$y_0 = a_{00}.$$

Let n = 1. Then

$$y_1 = a_{01}x + a_{11},$$

$$y_1' = a_{01},$$

$$y_1'' = y_1''' = y_1^{iv} = 0,$$

so that

$$4a_{01}x - 4(a_{01}x + a_{11}) = 0,$$

$$0 \cdot a_{01}x - 4a_{11} = 0.$$

Thus, a_{01} is arbitrary and $a_{11} = 0$. Hence,

$$y_1 = a_{01}x.$$

Let n = 2. Then

$$y_2 = a_{02}x^2 + a_{12}x + a_{22},$$

$$y_2' = 2a_{02}x + a_{12},$$

$$y_2'' = 2a_{02},$$

$$y_2''' = y_2^{iv} = 0,$$

so that

$$2(14x^2-6)a_{02} + 4x(2a_{02}x + a_{12}) - 36(a_{02}x^2 + a_{12}x + a_{22}) = 0,$$

$$0 \cdot a_{02}x^2 - 32a_{12}x - (12a_{02} + 36a_{22}) = 0.$$

Thus, a_{02} is arbitrary, $a_{12} = 0$ and $a_{22} = -(1/3)a_{02}$. Hence,

$$y_2 = a_{02}\left(x^2 - \tfrac{1}{3}\right).$$

Let n = 3. Then

$$y_3 = a_{03}x^3 + a_{13}x^2 + a_{23}x + a_{33},$$

$$y_3' = 3a_{03}x^2 + 2a_{13}x + a_{23},$$

$$y_3'' = 6a_{03}x + 2a_{13},$$

$$y_3''' = 6a_{03},$$

$$y_3^{iv} = 0,$$

so that

$$6(8x^3-8x)a_{03} + (14x^2-6)(6a_{03}x + 2a_{13}) + 4x(3a_{03}x^2$$

$$+ 2a_{13} + a_{23}) - 144(a_{03}x^3 + a_{13}x^2 + a_{23}x + a_{33}) = 0,$$

$$0 \cdot a_{03} x^3 - 108 a_{13} x^2 - (84 a_{03} + 140 a_{23}) x - (12 a_{13} + 144 a_{33}) = 0.$$

Thus, a_{03} is arbitrary, $a_{13} = 0$, $a_{23} = -(3/5)a_{03}$ and $a_{33} = 0$. Hence,

$$y_3 = a_{03}(x^3 - \tfrac{3}{5}x).$$

Let n = 4. Then

$$y_4 = a_{04} x^4 + a_{14} x^3 + a_{24} x^2 + a_{34} x + a_{44},$$

$$y_4' = 4a_{04} x^3 + 3a_{14} x^2 + 2a_{24} x + a_{34},$$

$$y_4'' = 12 a_{04} x^2 + 6 a_{14} x + 2 a_{24},$$

$$y_4''' = 24 a_{04} x + 6 a_{14},$$

$$y_4^{IV} = 24 a_{04},$$

so that

$$24(x^4 - 2x^2 + 1) a_{04} + (8x^3 - 8x)(24 a_{04} x + 6 a_{14})$$

$$+ (14x^2 - 6)(12 a_{04} x^2 + 6 a_{14} x + 2 a_{24}) + 4x(4 a_{04} x^3 + 3 a_{14} x^2$$

$$+ 2 a_{24} x + a_{34}) - 400(a_{04} x^4 + a_{14} x^3 + a_{24} x^2 + a_{34} x + a_{44}) = 0,$$

$$0 \cdot a_{04} x^4 - 256 a_{14} x^3 - (312 a_{04} + 364 a_{24}) x^2$$

$$- (84 a_{14} + 396 a_{34}) x + (24 a_{04} - 12 a_{24} - 400 a_{44}) = 0.$$

Thus a_{04} is arbitrary, $a_{14} = 0$, $a_{24} = -(6/7)a_{04}$, $a_{34} = 0$ and $a_{44} = (3/35)a_{04}$. Consequently,

$$y_4 = a_{04}(x^4 - \tfrac{6}{7}x^2 + \tfrac{3}{35}).$$

Further solutions y_n, $n = 5,6,7,\ldots$, may be obtained in a similar manner.

These polynomials form an orthogonal system over the fundamental interval $(-1,1)$ and are analogues of the Jacobi polynomials. Furthermore, if the a_{0n}, $n = 0,1,2,\ldots$, are chosen in such a manner that $y_n(1) = 1$, then the preceding solution set is seen to be precisely the Legendre polynomials.

Example 2

Let $g = 1$, $h = 0$, $\alpha = 0$ and $\beta = 1$. Then

$$\rho = x.$$

$$P = Ax^2(x-1)^2 = A(x^4 - 2x^3 + x^2).$$

$$Q = 2Ax(x-1)(5x-3) = A(10x^3 - 16x^2 + 6x).$$

$$R = B_1 x^2 + B_2 x + B_3,$$

where

$$R(0) = 6A = B_3,$$

$$R(1) = 2A = B_1 + B_2 + B_3,$$

so that $B_2 = -(B_1 + 4A)$ and $B_3 = 6A$. Thus,

$$R = B_1 x^2 - (B_1 + 4A)x + 6A.$$

$$S = (3B_1 - 60A)x - (2B_1 - 40A).$$

The differential equation is

$$A(x^4 - 2x^3 + x^2)y_n^{iv} + A(10x^3 - 16x^2 + 6x)y_n''' + [B_1 x^2 - (B_1 + 4A)x$$

$$+ 6A]y_n'' + [(3B_1 - 60A)x - (2B_1 - 40A)]y_n' + \lambda_n y_n = 0.$$

Solutions of the form

$$y_n = \sum_{j=0}^{n} a_{jn} x^{n-j}, \quad a_{0n} \neq 0,$$

will be obtained.

Consequently,

λ_n	$y_n = a_{0n}x^n + \cdots + a_{nn}, \quad a_{0n} \neq 0$
$(3B_1-60A)x-(2B_1-40A)$	$y_n' = na_{0n}x^{n-1} + \cdots + a_{(n-1)n}$
$B_1x^2-(B_1+4A)x+6A$	$y_n'' = n(n-1)a_{0n}x^{n-2} + \cdots + 2a_{(n-2)n}$
$A(10x^3-16x^2+6x)$	$y_n''' = n(n-1)(n-2)a_{0n}x^{n-3} + \cdots + 6a_{(n-3)n}$
$A(x^4-2x^3+x^2)$	$y_n^{iv} = n(n-1)(n-2)(n-3)a_{0n}x^{n-4} + \cdots + 24a_{(n-4)n}.$

The highest power of x that appears is n. The coefficient of x^n is

$$[\lambda_n + (3B_1-60A)n + B_1n(n-1) + 10An(n-1)(n-2) + An(n-1)(n-2)(n-3)]a_{0n} = 0.$$

Since $a_{0n} \neq 0$, then

$$\lambda_n = -n[3B_1 - 60A + B_1(n-1) + 10A(n-1)(n-2) + A(n-1)(n-2)(n-3)]$$

$$= -n[An^3 + 4An^2 + (B_1-19A)n + 2B_1 - 46A].$$

The choices of A and B_1 are arbitrary, with the exception that $B_1 \neq 20A$ since $S \equiv 0$ if $B_1 = 20A$. Hence, choose $A = 1$ and $B_1 = 23$ and thus simplify λ_n to

$$\lambda_n = -n(n^3+4n^2+4n) = -n^2(n+2)^2.$$

The differential equation can now be written as

$$(x^4-2x^3+x^2)y_n^{iv} + (10x^3-16x^2+6x)y_n''' + (23x^2-27x+6)y_n''$$

$$+ (9x-6)y_n' - n^2(n+2)^2y_n \quad 0.$$

For each choice of n, where n = 0,1,2,..., a polynomial solution of degree n will be obtained. The determinations of a few of these solutions follow.

Let n = 0. Then

$$y_0 = a_{00},$$

$$y_0' = y_0'' = y_0''' = y_0^{IV} = 0,$$

so that

$$0 \cdot a_{00} = 0.$$

Thus, a_{00} is arbitrary and

$$y_0 = a_{00}.$$

Let n = 1. Then

$$y_1 = a_{01}x + a_{11},$$

$$y_1' = a_{01},$$

$$y_1'' = y_1''' = y_1^{IV} = 0,$$

so that

$$(9x-6)a_{01} - 9(a_{01}x + a_{11}) = 0,$$

$$0 \cdot a_{01}x - (6a_{01} + 9a_{11}) = 0.$$

Thus, a_{01} is arbitrary and $a_{11} = -(2/3)a_{01}$. Consequently,

$$y_1 = a_{01}(x - \tfrac{2}{3}).$$

Let n = 2. Then

$$y_2 = a_{02}x^2 + a_{12}x + a_{22},$$

$$y_2' = 2a_{02}x + a_{12},$$

$$y_2'' = 2a_{02},$$

$$y_2''' = y_2^{IV} = 0,$$

so that

$$2(23x^2-27x+6)a_{02} + (9x-6)(2a_{02}x + a_{12}) - 64(a_{02}x^2 + a_{12}x + a_{22}) = 0,$$

$$0 \cdot a_{02}x^2 - (66a_{02} + 55a_{12})x + (12a_{02} - 6a_{12} - 64a_{22}) = 0.$$

Thus, a_{02} is arbitrary, $a_{12} = -(6/5)a_{02}$ and $a_{22} = (3/10)a_{02}$. Hence,

$$y_2 = a_{02}\left(x^2 - \tfrac{6}{5}x + \tfrac{3}{10}\right).$$

Let n = 3. Then

$$y_3 = a_{03}x^3 + a_{13}x^2 + a_{23}x + a_{33},$$

$$y_3' = 3a_{03}x^2 + 2a_{13}x + a_{23},$$

$$y_3'' = 6a_{03}x + 2a_{13},$$

$$y_3''' = 6a_{03},$$

$$y_3^{IV} = 0,$$

so that

$$6(10x^3-16x^2+6x)a_{03}+(23x^2-27x+6)(6a_{03}x+2a_{13})$$

$$+(9x-6)(3a_{03}x^2+2a_{13}x+a_{23})-225(a_{03}x^3+a_{13}x^2+a_{23}x+a_{33})=0,$$

$$0\cdot a_{03}x^3-(276a_{03}+161a_{13})x^2+(72a_{03}-66a_{13}-216a_{23})x$$

$$+(12a_{13}-6a_{23}-225a_{33})=0.$$

Thus, a_{03} is arbitrary, $a_{13} = -(12/7)a_{03}$, $a_{23} = (6/7)a_{03}$ and $a_{33} = -(4/35)a_{03}$. Consequently,

$$y_3 = a_{03}\left(x^3 - \tfrac{12}{7}x^2 + \tfrac{6}{7}x - \tfrac{4}{35}\right).$$

Further solutions y_n, $n = 4,5,6,\ldots$, may be obtained in a similar manner.

CHAPTER V

SOLUTIONS IN THE SEMI-INFINITE INTERVAL

The Semi-infinite Interval

The case of the fundamental interval (α, β) which extends to infinity in one direction will be discussed in this chapter. Two situations arise here, namely, the interval (α, ∞) and the interval $(-\infty, \alpha)$. It will be seen that both situations are fundamentally the same, and consequently, the discussion will be based on the interval (α, ∞), where α is finite.

The Weight Function $\rho(x)$

As in Chapter IV, the vanishing of $\rho(x)$, $\rho'(x)$ and $\rho''(x)$ at the finite value α must be considered if $\rho(x)$ is to play its part in the satisfaction of the five orthogonality conditions.

This case, however, differs from that of the finite interval in that the interval extends to infinity in one direction. For the sake of convenience, the condition that

$$\lim_{x \to \infty} f(x) = 0$$

will henceforth be denoted by the statement "$f(x) = 0$ at ∞".

The fact that P is a polynomial prohibits the vanishing of P as $x \to \infty$. Consequently, in orthogonality condition (i), $\rho(x)$ must vanish as $x \to \infty$.

Orthogonality condition (ii) will require that $\rho'(x)$ vanish as $x \to \infty$. Orthogonality condition (iii) forces $\rho''(x)$ to vanish as $x \to \infty$.

Consideration of the identities in orthogonality conditions (iv) and (v) indicates that ρ again must be a factor of the right members of both of these identities. Consequently, if ρ', ρ'' and ρ''' are each expressible as the product of ρ and a rational function, where the denominator of the rational function divides the numerator of the term in which it appears, then ρ will be such that the orthogonality conditions are satisfied.

Consider a choice of

$$\rho = e^{-hx}(x-\alpha)^g, \tag{5.1}$$

where g and h are both constants, $h > 0$ and g is a real number. Successive differentiations of equation (5.1) will give:

$$\rho' = -he^{-hx}(x-\alpha)^g + ge^{-hx}(x-\alpha)^{g-1} = e^{-hx}(x-\alpha)^g\left[\frac{-h(x-\alpha)+g}{x-\alpha}\right], \tag{5.2}$$

$$\rho'' = h^2e^{-hx}(x-\alpha)^g - 2ghe^{-hx}(x-\alpha)^{g-1} + g(g-1)e^{-hx}(x-\alpha)^{g-2} \tag{5.3}$$

$$= e^{-hx}(x-\alpha)^g\left[\frac{h^2(x-\alpha)^2 - 2gh(x-\alpha) + g(g-1)}{(x-\alpha)^2}\right],$$

$$\rho''' = -h^3e^{-hx}(x-\alpha)^g + 3gh^2e^{-hx}(x-\alpha)^{g-1} - 3g(g-1)he^{-hx}(x-\alpha)^{g-2} \tag{5.4}$$

$$+ g(g-1)(g-2)e^{-hx}(x-\alpha)^{g-3}$$

$$= e^{-hx}(x-\alpha)^g\left[\frac{-h^3(x-\alpha)^3 + 3gh^2(x-\alpha)^2 - 3g(g-1)h(x-\alpha) + g(g-1)(g-2)}{(x-\alpha)^3}\right].$$

So, with the choice $h > 0$, and the proper restrictions on g, $\rho = e^{-hx}(x-\alpha)^g$ will be sufficient to satisfy the orthogonality conditions.

The Leading Coefficient $P(x)$

The function e^{-hx} and its successive derivatives do not vanish at any finite value $x = \alpha$. A reappraisal of the contradictions established in Chapter IV indicates the invalidity of $P(x) = 0$ having α as a root of multiplicity less than two. Replacement of the factor $(x-\beta)^h$, in the $\rho(x)$ of Chapter IV, by e^{-hx} will effect these same contradictions. This can be seen by noting that neither $(x-\beta)^h$ and its derivatives nor e^{-hx} and its derivatives will ever vanish at the finite value $x = \alpha$.

Consequently, $P(x) = (x-\alpha)^2V(x)$, where $V(x)$ is a polynomial of degree two, at most.

Application of orthogonality conditions. With the choice of $\rho = e^{-hx}(x-\alpha)^g$, where $h > 0$, and with $P = (x-\alpha)^2V$, orthogonality condition (v) gives

$$(5.5)\qquad e^{-hx}(x-\alpha)^g Q \equiv 2[e^{-hx}(x-\alpha)^{g+2}V]'$$

$$\equiv 2[-he^{-hx}(x-\alpha)^{g+2}V + (g+2)e^{-hx}(x-\alpha)^{g+1}V + e^{-hx}(x-\alpha)^{g+2}V'],$$

or

$$(5.6)\qquad Q \equiv 2(x-\alpha)[-h(x-\alpha)V + (g+2)V + (x-\alpha)V'].$$

Let $V = C_1x^2 + C_2x + C_3$. Then

$$(5.7)\qquad Q \equiv 2(x-\alpha)[-h(x-\alpha)(C_1x^2+C_2x+C_3) + (g+2)(C_1x^2+C_2x+C_3) + (x-\alpha)(2C_1x+C_2)].$$

The coefficient Q is a polynomial of degree three, at most, so the coefficient of x^4 in the right member of equation (5.7) must vanish. Hence,

$$-2hC_1 = 0, \tag{5.8}$$

but since $h \neq 0$, then $C_1 = 0$ and

$$V = C_2 x + C_3. \tag{5.9}$$

Application of orthogonality condition (iv) yields

$$\begin{aligned} e^{-hx}(x-\alpha)^g S &\equiv \left[e^{-hx}(x-\alpha)^g R\right]' - \left[e^{-hx}(x-\alpha)^{g+2} V\right]''' \\ &\equiv -he^{-hx}(x-\alpha)^g R + ge^{-hx}(x-\alpha)^{g-1} R \\ &\quad + e^{-hx}(x-\alpha)^g R' - \left[-he^{-hx}(x-\alpha)^{g+2} V\right. \\ &\quad \left. + (g+2)e^{-hx}(x-\alpha)^{g+1} V + e^{-hx}(x-\alpha)^{g+2} V'\right]'' \\ &\equiv -he^{-hx}(x-\alpha)^g R + ge^{-hx}(x-\alpha)^{g-1} R \\ &\quad + e^{-hx}(x-\alpha)^g R' - \left[h^2 e^{-hx}(x-\alpha)^{g+2} V\right. \\ &\quad - 2h(g+2)e^{-hx}(x-\alpha)^{g+1} V - 2he^{-hx}(x-\alpha)^{g+2} V' \\ &\quad \left. + (g+1)(g+2)e^{-hx}(x-\alpha)^g V + 2(g+2)e^{-hx}(x-\alpha)^{g+1} V'\right]' \\ &\equiv -he^{-hx}(x-\alpha)^g R + ge^{-hx}(x-\alpha)^{g-1} R \\ &\quad + e^{-hx}(x-\alpha)^g R' - \left[-h^3 e^{-hx}(x-\alpha)^{g+2} V\right. \\ &\quad + 3h^2(g+2)e^{-hx}(x-\alpha)^{g+1} V + 3h^2 e^{-hx}(x-\alpha)^{g+2} V' \\ &\quad - 3h(g+1)(g+2)e^{-hx}(x-\alpha)^g V - 6h(g+2)e^{-hx}(x-\alpha)^{g+1} V' \\ &\quad \left. + g(g+1)(g+2)e^{-hx}(x-\alpha)^{g-1} V + 3(g+1)(g+2)e^{-hx}(x-\alpha)^g V'\right], \end{aligned} \tag{5.10}$$

or

$$(5.11)\quad S \equiv (x-\alpha)^{-1}[-h(x-\alpha)R + gR + (x-\alpha)R' + h^3(x-\alpha)^3V - 3h^2(g+2)(x-\alpha)^2V$$

$$-3h^2(x-\alpha)^3V' + 3h(g+1)(g+2)(x-\alpha)V + 6h(g+2)(x-\alpha)^2V'$$

$$-g(g+1)(g+2)V - 3(g+1)(g+2)(x-\alpha)V'].$$

Since S is a polynomial, then $(x-\alpha)$ divides the expression in brackets in equation (5.11). If $(x-\alpha)$ divides this expression, then

$$(5.12)\qquad gR(\alpha) - g(g+1)(g+2)V(\alpha) = 0,$$

and equation (5.11) becomes

$$(5.13)\quad S \equiv \frac{gR - g(g+1)(g+2)V}{x-\alpha} - hR + R' + h^3(x-\alpha)^2V - 3h^2(g+2)(x-\alpha)V$$

$$-3h^2(x-\alpha)^2V' + 3h(g+1)(g+2)V + 6h(g+2)(x-\alpha)V' - 3(g+1)(g+2)V'.$$

From equation (5.9), $V = C_2x + C_3$, and thus $V' = C_2$. Let $R = B_1x^2 + B_2x + B_3$, and then equation (5.13) becomes

$$(5.14)\quad S \equiv \frac{g(B_1x^2 + B_2x + B_3) - g(g+1)(g+2)(C_2x + C_3)}{x-\alpha} - h(B_1x^2 + B_2x + B_3)$$

$$+(2B_1x + B_2) + h^3(x-\alpha)^2(C_2x + C_3) - 3h^2(g+2)(x-\alpha)(C_2x + C_3)$$

$$-3h^2(x-\alpha)^2C_2 + 3h(g+1)(g+2)(C_2x + C_3) + 6h(g+2)(x-\alpha)C_2$$

$$-3(g+1)(g+2)C_2.$$

Since S is a first degree polynomial, the coefficients of x^3 and x^2, in the right member of equation (5.14), must vanish. If the coefficient of x^3 vanishes, then

(5.15) $$h^3 C_2 = 0,$$

and since $h \neq 0$, then $C_2 = 0$, and thus

(5.16) $$V = C_3 .$$

Also, if the coefficient of x^2 vanishes, then

(5.17) $$-hB_1 + h^3 C_3 = 0,$$

so that

(5.18) $$B_1 = h^2 C_3 .$$

Now

(5.19) $$R = h^2 C_3 x^2 + B_2 x + B_3 ,$$

and equation (5.14) can be written as

(5.20) $$S \equiv \frac{g(h^2C_3x^2 + B_2x + B_3) - g(g+1)(g+2)C_3}{x-\alpha} - h(B_2x + B_3)$$

$$+ (2h^2C_3x + B_2) - 2h^3C_3\alpha x + h^3\alpha^2 C_3$$

$$-3h^2(g+2)(x-\alpha)C_3 + 3h(g+1)(g+2)C_3 .$$

Apply orthogonality condition (i) to obtain

(5.21) $$\rho P = c_3 e^{-hx}(x-\alpha)^{g+2} = 0$$

at α and ∞. Since $h > 0$, then $\rho P = 0$ at ∞. For ρP to vanish at $x = \alpha$, then

(5.22) $$g > -2.$$

Orthogonality condition (ii) gives

(5.23) $$(\rho P)' = c_3 \left[-h e^{-hx}(x-\alpha)^{g+2} + (g+2)e^{-hx}(x-\alpha)^{g+1}\right]$$

$$= c_3 e^{-hx}(x-\alpha)^{g+1}\left[-h(x-\alpha) + (g+2)\right].$$

Again, $h > 0$ implies $(\rho P)' = 0$ at ∞, and if $g > -1$, then $(\rho P)' = 0$ at α. If $-2 < g \leqq -1$, then $g+2 = 0$, which is not possible. Hence,

(5.24) $$g > -1.$$

The application of orthogonality condition (iii) yields

(5.25) $$\rho R - (\rho P)'' = e^{-hx}(x-\alpha)^{g} R - c_3\left[h^2 e^{-hx}(x-\alpha)^{g+2}\right.$$

$$\left. - 2h(g+2)e^{-hx}(x-\alpha)^{g+1} + (g+1)(g+2)e^{-hx}(x-\alpha)^{g}\right]$$

$$= e^{-hx}(x-\alpha)^{g}\left[R - c_3 h^2 (x-\alpha)^2 + 2c_3 h(g+2)(x-\alpha)\right.$$

$$\left. - c_3(g+1)(g+2)\right] = 0$$

at α and ∞. Here $h > 0$ implies that $\rho R - (\rho P)'' = 0$ at ∞, and if $g > 0$, then $\rho R - (\rho P)'' = 0$ at α. If $-1 < g \leqq 0$, then

(5.26) $$R(\alpha) - c_3(g+1)(g+2) = 0.$$

From equation (5.12), if $g \neq 0$, then $R(\alpha) - C_3(g+1)(g+2) = 0$. Hence,

(5.27) $$R(\alpha) = C_3(g+1)(g+2), \quad g > -1.$$

The Interval $(-\infty, \alpha)$

The foregoing development used the fact that $h \neq 0$. If ρ is chosen as $\rho = e^{-hx}(x-\alpha)^g$, where $h < 0$ (rather than $h > 0$), and $g > -1$, then the preceding development will be reproduced and will yield comparable results for the interval $(-\infty, \alpha)$. When this interval is under consideration, the condition that

$$\lim_{x \to -\infty} f(x) = 0$$

will be denoted by the statement "$f(x) = 0$ at $-\infty$".

Summary

The choice of $\rho = e^{-hx}(x-\alpha)^g$, where $h > 0$ (or $h < 0$), and $g > -1$, forces P to be of the form

$$P = C_3(x-\alpha)^2.$$

From equation (5.6),

$$Q = 2(x-\alpha)\left[-hC_3(x-\alpha) + C_3(g+2)\right].$$

The coefficient R is of the form

$$R = B_1x^2 + B_2x + B_3,$$

where, from equation (5.18),

$$B_1 = h^2C_3,$$

and, from equation (5.27),

$$R(\alpha) = c_3 (g+1)(g+2).$$

From equation (5.20),

$$S \equiv \frac{g(B_1 x^2 + B_2 x + B_3) - c_3 g(g+1)(g+2)}{x-\alpha} - h(B_2 x + B_3)$$

$$+ (2B_1 x + B_2) - 2h^3 c_3 \alpha x + h^3 \alpha^2 c_3$$

$$- 3h^2 c_3 (g+2)(x-\alpha) + 3h c_3 (g+1)(g+2).$$

These choices of P, Q, R and S will satisfy all five orthogonality conditions for the interval (α, ∞) $[$or $(-\infty, \alpha)]$, with respect to the weight function $\rho = e^{-hx}(x-\alpha)^g$.

Examples of the Semi-infinite Interval

Example 1

Let h = 1, g = 0, $\alpha = 0$ and consider the interval (α, ∞). Then

$$\rho = e^{-x}.$$

$$P = c_3 x^2.$$

$$Q = 2x(2c_3 - c_3 x) = c_3(4x - 2x^2).$$

$$R = B_1 x^2 + B_2 x + B_3,$$

where

$$B_1 = c_3,$$

$$R(0) = 2c_3 = B_3,$$

so that

$$R = 2c_3 + B_2 x + c_3 x^2.$$

$$S = (B_2 + 4c_3) - (B_2 + 4c_3)x.$$

The differential equation is

$$c_3 x^2 y_n^{\mathrm{IV}} + c_3(4x - 2x^2) y_n''' + (2c_3 + B_2 x + c_3 x^2) y_n''$$

$$+ [(B_2 + 4c_3) - (B_2 + 4c_3)x] y_n' + \lambda_n y_n = 0.$$

Solutions of the form

$$y_n = \sum_{j=0}^{n} a_{jn} x^{n-j}, \quad a_{0n} \neq 0,$$

will be obtained.

Consequently,

λ_n	$y_n = a_{0n}x^n + \cdots + a_{nn},\ a_{0n} \neq 0$
$(B_2+4C_3)-(B_2+4C_3)x$	$y_n' = na_{0n}x^{n-1} + \cdots + a_{(n-1)n}$
$2C_3 + B_2x + C_3x^2$	$y_n'' = n(n-1)a_{0n}x^{n-2} + \cdots + 2a_{(n-2)n}$
$C_3(4x-2x^2)$	$y_n''' = n(n-1)(n-2)a_{0n}x^{n-3} + \cdots + 6a_{(n-3)n}$
C_3x^2	$y_n^{iv} = n(n-1)(n-2)(n-3)a_{0n}x^{n-4} + \cdots + 24a_{(n-4)n}.$

The highest power of x that appears is n. The coefficient of x^n is

$$[\lambda_n - (B_2+4C_3)n + C_3 n(n-1)]a_{0n} = 0.$$

Since $a_{0n} \neq 0$, then

$$\lambda_n = -n[-B_2 - 4C_3 + C_3(n-1)],$$

$$= -n(C_3 n - B_2 - 5C_3).$$

The choices of C_3 and B_2 are arbitrary, with the exception that $B_2 \neq -4C_3$ since $S \equiv 0$ if $B_2 = -4C_3$. Hence, choose $C_3 = 1$ and $B_2 = -5$ and thus simplify λ_n to

$$\lambda_n = -n^2.$$

The differential equation can now be written as

$$x^2 y_n^{iv} + (4x-2x^2)y_n''' + (2-5x+x^2)y_n'' + (-1+x)y_n' - n^2 y_n = 0.$$

For each choice of n, where n = 0,1,2,..., a polynomial solution of degree n will be obtained. The determinations of a few of these solutions follow.

Let $n = 0$. Then

$$y_0 = a_{00},$$

$$y_0' = y_0'' = y_0''' = y_0^{iv} = 0,$$

so that

$$0 \cdot a_{00} = 0.$$

Thus, a_{00} is arbitrary and

$$y_0 = a_{00}.$$

Let $n = 1$. Then

$$y_1 = a_{01}x + a_{11},$$

$$y_1' = a_{01},$$

$$y_1'' = y_1''' = y_1^{iv} = 0,$$

so that

$$(-1 + x)a_{01} - (a_{01}x + a_{11}) = 0,$$

$$0 \cdot a_{0n}x - (a_{01} + a_{11}) = 0.$$

Thus, a_{01} is arbitrary and $a_{11} = -a_{01}$. Hence,

$$y_1 = a_{01}(x - 1).$$

Let $n = 2$. Then

$$y_2 = a_{02}x^2 + a_{12}x + a_{22},$$

$$y_2' = 2a_{02}x + a_{12},$$

$$y_2'' = 2a_{02},$$

$$y_2''' = y_2^{\mathrm{iv}} = 0,$$

so that

$$2(2-5x+x^2)a_{02} + (-1+x)(2a_{02}x + a_{12}) - 4(a_{02}x^2 + a_{12}x + a_{22}) = 0,$$

$$0 \cdot a_{02}x^2 - (12a_{02} + 3a_{12})x + (4a_{02} - a_{12} - 4a_{22}) = 0.$$

Thus, a_{02} is arbitrary, $a_{12} = -4a_{02}$ and $a_{22} = 2a_{02}$. Consequently,

$$y_2 = a_{02}(x^2 - 4x + 2).$$

Let Let $n = 3$. Then

$$y_3 = a_{03}x^3 + a_{13}x^2 + a_{23}x + a_{33},$$

$$y_3' = 3a_{03}x^2 + 2a_{13}x + a_{23},$$

$$y_3'' = 6a_{03}x + 2a_{13},$$

$$y_3''' = 6a_{03},$$

$$y_3^{\mathrm{iv}} = 0,$$

so that

$$6(4x - 2x^2)a_{03} + (2-5x+x^2)(6a_{03}x + 2a_{13})$$

$$+ (-1+x)(3a_{03}x^2 + 2a_{13}x + a_{23}) - 9(a_{03}x^3 + a_{13}x^2 + a_{23}x + a_{33}) = 0,$$

$$0 \cdot a_{03} x^3 - (45a_{03} + 5a_{13})x^2 + (36a_{03} - 12a_{13} - 8a_{23})x + (4a_{13} - a_{23} - 9a_{33}) = 0.$$

Thus, a_{03} is arbitrary, $a_{13} = -9a_{03}$, $a_{23} = 18a_{03}$ and $a_{33} = -6a_{03}$.
Consequently,

$$y_3 = a_{03}(x^3 - 9x^2 + 18x - 6).$$

Further solutions y_n, $n = 4,5,6,\ldots$, can be obtained in a similar manner.

The preceding set of polynomial solutions are orthogonal over the fundamental interval $(0,\infty)$, with respect to the weight function $\rho = e^{-x}$, and thus are considered as analogues to a set of Laguerre polynomials.

Example 2

Let $h = -1$, $g = 1$, $\alpha = 1$ and consider the interval $(-\infty, \alpha)$. Then

$$\rho = (x-1)e^{x}.$$

$$P = C_3(x-1)^2 = C_3(x^2-2x+1).$$

$$Q = 2(x-1)[C_3(x-1)+3C_3] = C_3(2x^2+2x-4).$$

$$R = B_1x^2 + B_2x + B_3,$$

where

$$B_1 = C_3,$$

$$R(1) = 6C_3 = B_1 + B_2 + B_3,$$

so that $B_1 = C_3$ and $B_3 = 5C_3 - B_2$. Then

$$R = C_3x^2 + B_2x + (5C_3 - B_2).$$

$$S = (B_2-4C_3)x + (B_2-4C_3).$$

The differential equation is

$$C_3(x^2-2x+1)y_n^{iv} + C_3(2x^2+2x-4)y_n''' + [C_3x^2 + B_2x$$

$$+ (5C_3-B_2)]y_n'' + [(B_2-4C_3)x + (B_2-4C_3)]y_n' + \lambda_n y_n = 0.$$

Solutions of the form

$$y_n = \sum_{j=0}^{n} a_{jn}x^{n-j}, \quad a_{0n} \neq 0,$$

will be obtained.

Consequently,

λ_n	$y_n = a_{0n}x^n + \cdots + a_{nn}, \quad a_{0n} \neq 0$
$(B_2 - 4C_3)x + (B_2 - 4C_3)$	$y_n' = na_{0n}x^{n-1} + \cdots + a_{(n-1)n}$
$C_3x^2 + B_2x + (5C_3 - B_2)$	$y_n'' = n(n-1)a_{0n}x^{n-2} + \cdots + 2a_{(n-2)n}$
$C_3(2x^2 + 2x - 4)$	$y_n''' = n(n-1)(n-2)a_{0n}x^{n-3} + \cdots + 6a_{(n-3)n}$
$C_3(x^2 - 2x + 1)$	$y_n^{iv} = n(n-1)(n-2)(n-3)a_{0n}x^{n-4} + \cdots + 24a_{(n-4)n}.$

The highest power of x that appears is n. The coefficient of x^n is

$$[\lambda_n + (B_2 - 4C_3)n + C_3n(n-1)]a_{0n} = 0.$$

Since $a_{0n} \neq 0$, then

$$\lambda_n = -n[B_2 - 4C_3 + C_3(n-1)],$$

$$= -n(C_3n + B_2 - 5C_3).$$

The choices of C_3 and B_2 are arbitrary, with the exception that $B_2 \neq 4C_3$, since $S \equiv 0$ if $B_2 = 4C_3$. Hence, choose $C_3 = 1$ and $B_2 = 5$ and thus simplify λ_n to

$$\lambda_n = -n^2.$$

The differential equation can now be written as

$$(x^2 - 2x + 1)y_n^{iv} + (2x^2 + 2x - 4)y_n''' + (x^2 + 5x)y_n'' + (x+1)y_n' - n^2y_n = 0.$$

For each choice of n, where n = 0,1,2,..., a polynomial solution of degree n will be obtained. The determinations of a few of these solutions follow.

Let n = 0. Then

$$y_0 = a_{00},$$

$$y_0' = y_0'' = y_0''' = y_0^{iv} = 0,$$

so that

$$0 \cdot a_{00} = 0.$$

Thus, a_{00} is arbitrary and

$$y_0 = a_{00}.$$

Let n = 1. Then

$$y_1 = a_{01}x + a_{11},$$

$$y_1' = a_{01},$$

$$y_1'' = y_1''' = y_1^{iv} = 0,$$

so that

$$(x+1)a_{01} - (a_{01}x + a_{11}) = 0,$$

$$0 \cdot a_{01}x + (a_{01} - a_{11}) = 0.$$

Thus, a_{01} is arbitrary and $a_{11} = a_{01}$. Consequently,

$$y_1 = a_{01}(x+1).$$

Let n = 2. Then

$$y_2 = a_{02}x^2 + a_{12}x + a_{22},$$

$$y_2' = 2a_{02}x + a_{12},$$

$$y_2'' = 2a_{02},$$

$$y_2''' = y_2^{iv} = 0,$$

so that

$$2(x^2+5x)a_{02} + (x+1)(2a_{02}x + a_{12}) - 4(a_{02}x^2 + a_{12}x + a_{22}) = 0,$$

$$0 \cdot a_{02}x^2 + (12a_{02} - 3a_{12})x + (a_{12} - 4a_{22}) = 0.$$

Thus, a_{02} is arbitrary, $a_{12} = 4a_{02}$ and $a_{22} = a_{02}$. Consequently,

$$y_2 = a_{02}(x^2 + 4x + 1).$$

Let n = 3. Then

$$y_3 = a_{03}x^3 + a_{13}x^2 + a_{23}x + a_{33},$$

$$y_3' = 3a_{03}x^2 + 2a_{13}x + a_{23},$$

$$y_3'' = 6a_{03}x + 2a_{13},$$

$$y_3''' = 6a_{03},$$

$$y_3^{iv} = 0,$$

so that

$$6(2x^2+2x-4)a_{03} + (x^2+5x)(6a_{03}x + 2a_{13})$$

$$+ (x+1)(3a_{03}x^2 + 2a_{13}x + a_{23}) - 9(a_{03}x^3 + a_{13}x^2 + a_{23}x + a_{33}) = 0,$$

$$0\cdot a_{03}x^3+(45a_{03}-5a_{13})x^2+(12a_{03}+12a_{13}-8a_{23})x$$

$$-(24a_{03}-a_{23}+9a_{33})=0.$$

Thus, a_{03} is arbitrary, $a_{13} = 9a_{03}$, $a_{23} = 15a_{03}$ and $a_{33} = -a_{03}$. Hence,

$$y_3 = a_{03}(x^3+9x^2+15x-1).$$

Further solutions y_n, $n = 4,5,6,\ldots$, can be obtained in a similar manner.

CHAPTER VI

SOLUTIONS IN THE INFINITE INTERVAL

The Infinite Interval

The fundamental interval (α,β) may extend in both directions to infinity. If this is the case, then a choice of $\rho(x)$ will have to be made that is somewhat different from the $\rho(x)$ of Chapters IV and V. As in these previous chapters, the choice of $\rho(x)$ will determine the basic forms of P, Q, R and S of differential equation (1.17).

The Weight Function $\rho(x)$

The polynomial nature of the coefficients P, Q, R and S of the differential equation (1.17) prohibits their vanishing at $\pm\infty$. An inspection of orthogonality conditions (i), (ii) and (iii) will then show that ρ, ρ' and ρ'' must each vanish at $\pm\infty$.

The identities in orthogonality conditions (iv) and (v) will again require that ρ', ρ'' and ρ''' be expressed as products of ρ and a rational function, as in the preceding cases for intervals with at least one finite end point. Also, the denominators of these rational functions must divide the numerators of the terms in which they appear.

Consider a choice of

(6.1) $$\rho = e^{-hx^2}, \ h > 0.$$

Successive differentiations of equation (6.1) give:

$$\rho' = -2hx\,e^{-hx^2}, \tag{6.2}$$

$$\rho'' = 4h^2x^2e^{-hx^2} - 2h\,e^{-hx^2} \tag{6.3}$$

$$= 2h(2hx^2-1)e^{-hx^2},$$

$$\rho''' = -8h^3x^3e^{-hx^2} + 12h^2x\,e^{-hx^2} \tag{6.4}$$

$$= 4h^2x(-2hx^2+3)e^{-hx^2}.$$

As the preceding equations indicate, the choice of $\rho = e^{-hx^2}$, $h > 0$, is sufficient to satisfy the orthogonality conditions.

Application of the orthogonality conditions. In this case, the initial use of orthogonality condition (iv) is revealing. Since is chosen as $\rho = e^{-hx^2}$, condition (iv) states

$$e^{-hx^2}S \equiv (e^{-hx^2}R)' - (e^{-hx^2}P)''' \tag{6.5}$$

$$\equiv -2hx e^{-hx^2}R + e^{-hx^2}R' - (-2hx e^{-hx^2}P + e^{-hx^2}P')''$$

$$\equiv -2hx e^{-hx^2}R + e^{-hx^2}R' - (-2he^{-hx^2}P$$

$$+ 4h^2x^2e^{-hx^2}P - 4hx e^{-hx^2}P' + e^{-hx^2}P'')'$$

$$\equiv -2hx e^{-hx^2}R + e^{-hx^2}R' - (12h^2x\,e^{-hx^2}P$$

$$-6he^{-hx^2}P' - 8h^3x^3e^{-hx^2}P + 12h^2x^2e^{-hx^2}P'$$

$$-6hx\,e^{-hx^2}P'' + e^{-hx^2}P''').$$

Let $P = A_1x^4 + A_2x^3 + A_3x^2 + A_4x + A_5$ and let $R = B_1x^2 + B_2x + B_3$. Solving equation (6.5) for S gives

$$S \equiv -2hx(B_1x^2 + B_2x + B_3) + (2B_1x + B_2) - 12h^2x(A_1x^4 + A_2x^3 + A_3x^2 + A_4x + A_5) + 6h(4A_1x^3 + 3A_2x^2 + 2A_3x + A_4) + 8h^3x^3(A_1x^4 + A_2x^3 + A_3x^2 + A_4x + A_5) - 12h^2x^2(4A_1x^3 + 3A_2x^2 + 2A_3x + A_4) + 6hx(12A_1x^2 + 6A_2x + 2A_3) - (24A_1x + 6A_2). \tag{6.6}$$

As S is a polynomial of degree one, the coefficients of x^7, x^6, x^5, x^4, x^3 and x^2 must vanish in the right member of equation (6.6).

The coefficient of x^7 is $8h^3A_1$. Since $h > 0$, this implies

$$A_1 = 0. \tag{6.7}$$

The coefficient of x^6 is $8h^3A_2$, and with $h > 0$, then

$$A_2 = 0. \tag{6.8}$$

The coefficient of x^5 is $8h^3A_3$, and thus

$$A_3 = 0. \tag{6.9}$$

The coefficient of x^4 is $8h^3A_4$, and so

$$A_4 = 0. \tag{6.10}$$

The coefficient of x^3 is $-2hB_1 + 8h^3A_5$, so that

$$B_1 = 4h^2A_5. \tag{6.11}$$

The coefficient of x^2 is $-2hB_2$, thus

(6.12) $$B_2 = 0.$$

Consequently, it is seen that $P = A_5$ and $R = 4h^2A_5x^2 + B_3$. Equation (6.6) can now be written as

(6.13) $$S \equiv -2hB_3x - 4h^2A_5x.$$

Application of orthogonality condition (i) gives

(6.14) $$\rho P = e^{-hx^2}A_5 = 0$$

at $\pm\infty$.

Orthogonality condition (ii) states

(6.15) $$(\rho P)' = -2hA_5xe^{-hx^2} = 0$$

at $\pm\infty$.

Consideration of orthogonality condition (iii) shows

(6.16) $$\rho R - (\rho P)'' = e^{-hx^2}R - (-2he^{-hx^2}P + 4h^2x^2e^{-hx^2}P)$$

$$= e^{-hx^2}(R + 2hP - 4h^2x^2P) = 0$$

at $\pm\infty$.

The only remaining orthogonality condition is (v). Its application yields

(6.17) $$e^{-hx^2}Q \equiv -4hA_5xe^{-hx^2},$$

and then

(6.18) $$Q \equiv -4hA_5x.$$

Summary

The choice of $\rho = e^{-hx^2}$, $h > 0$, forces P to be of the form

$$P = A_5.$$

From equation (6.18),

$$Q = -4hA_5 x.$$

The coefficient R is of the form

$$R = B_1 x^2 + B_3,$$

where, from equation (6.11),

$$B_1 = 4h^2 A_5.$$

From equation (6.13),

$$S = -2h(B_3 + 2hA_5)x.$$

These choices of P, Q, R and S will satisfy all five orthogonality conditions for the interval $(-\infty, \infty)$, with respect to the weight function $\rho = e^{-hx^2}$.

Example of the Infinite Interval

Let h = 1 and consider the interval $(-\infty, \infty)$. Then

$$\rho = e^{-x^2}.$$

$$P = A_5.$$

$$Q = -4A_5 X.$$

$$R = 4A_5 X^2 + B_3.$$

$$S = -2(B_3 + 2A_5)X.$$

The differential equation is

$$A_5 y_n^{iv} - 4A_5 X y_n''' + (4A_5 X^2 + B_3) y_n'' - 2(B_3 + 2A_5) X y_n' + \lambda_n y_n = 0.$$

Solutions of the form

$$y_n = \sum_{j=0}^{n} a_{jn} X^{n-j}, \quad a_{0n} \neq 0,$$

will be obtained.

Consequently,

$$\begin{array}{r|l}
\lambda_n & y_n = a_{0n} X^n + \cdots + a_{nn}, \quad a_{0n} \neq 0 \\
(-2B_3 - 4A_5)X & y_n' = n a_{0n} X^{n-1} + \cdots + a_{(n-1)n} \\
4A_5 X^2 + B_3 & y_n'' = n(n-1) a_{0n} X^{n-2} + \cdots + 2a_{(n-2)n} \\
-4A_5 X & y_n''' = n(n-1)(n-2) a_{0n} X^{n-3} + \cdots + 6a_{(n-3)n} \\
A_5 & y_n^{iv} = n(n-1)(n-2)(n-3) a_{0n} X^{n-4} + \cdots + 24a_{(n-4)n}. \\
\hline
\end{array}$$

The highest power of x that appears is n. The coefficient of x^n is

$$[\lambda_n - (2B_3 + 4A_5)n + 4A_5 n(n-1)]\, a_{0n} = 0.$$

Since $a_{0n} \neq 0$, then

$$\lambda_n = -n(-2B_3 - 4A_5 + 4A_5 n - 4A_5)$$

$$= -n(4A_5 n - 2B_3 - 8A_5).$$

The choices of A_5 and B_3 are arbitrary, with the exception that $B_3 \neq -2A_5$ since $S \equiv 0$ if $B_3 = -2A_5$. Hence, choose $A_5 = 1$ and $B_3 = -4$ and thus simplify λ_n to

$$\lambda_n = -4n^2.$$

The differential equation can now be written as

$$y_n^{iv} - 4Xy_n''' + (4X^2 - 4)y_n'' + 4Xy_n' - 4n^2 y_n = 0.$$

For each choice of n, where n = 0,1,2,..., a polynomial solution of degree n will be obtained. The determinations of a few of these solutions follow.

Let n = 0. Then

$$y_0 = a_{00},$$

$$y_0' = y_0'' = y_0''' = y_0^{iv} = 0,$$

so that

$$0 \cdot a_{00} = 0.$$

Thus, a_{00} is arbitrary and

$$y_0 = a_{00}.$$

Let n = 1. Then

$$y_1 = a_{01}x + a_{11},$$

$$y_1' = a_{01},$$

$$y_1'' = y_1''' = y_1^{iv} = 0,$$

so that

$$4a_{01}x - 4(a_{01}x + a_{11}) = 0,$$

$$0 \cdot a_{01}x - 4a_{11} = 0.$$

Thus, a_{01} is arbitrary and $a_{11} = 0$. Consequently,

$$y_1 = a_{01}x.$$

Let n = 2. Then

$$y_2 = a_{02}x^2 + a_{12}x + a_{22},$$

$$y_2' = 2a_{02}x + a_{12},$$

$$y_2'' = 2a_{02},$$

$$y_2''' = y_2^{iv} = 0,$$

so that

$$2(4x^2-4)a_{02} + 4x(2a_{02}x + a_{12}) - 16(a_{02}x^2 + a_{12}x + a_{22}) = 0,$$

$$0 \cdot a_{02}x^2 - 12a_{12}x - (8a_{02} + 16a_{22}) = 0.$$

Thus, a_{02} is arbitrary, $a_{12} = 0$ and $a_{22} = -(1/2)a_{02}$. Consequently,

$$y_2 = a_{02}(x^2 - \tfrac{1}{2}).$$

<u>Let n = 3</u>. Then

$$y_3 = a_{03}x^3 + a_{13}x^2 + a_{23}x + a_{33},$$

$$y_3' = 3a_{03}x^2 + 2a_{13}x + a_{23},$$

$$y_3'' = 6a_{03}x + 2a_{13},$$

$$y_3''' = 6a_{03},$$

$$y_3^{iv} = 0,$$

so that

$$-24a_{03}x + (4x^2-4)(6a_{03}x + 2a_{13}) + 4x(3a_{03}x^2$$

$$+ 2a_{13}x + a_{23}) - 36(a_{03}x^3 + a_{13}x^2 + a_{23}x + a_{33}) = 0,$$

$$0 \cdot a_{03}x^3 - 20a_{13}x^2 - (48a_{03} + 32a_{23})x - (8a_{13} + 36a_{33}) = 0.$$

Thus, a_{03} is arbitrary, $a_{13} = 0$, $a_{23} = -(3/2)a_{03}$ and $a_{33} = 0$. Hence,

$$y_3 = a_{03}(x^3 - \tfrac{3}{2}x).$$

<u>Let n = 4</u>. Then

$$y_4 = a_{04}x^4 + a_{14}x^3 + a_{24}x^2 + a_{34}x + a_{44},$$

$$y_4' = 4a_{04}x^3 + 3a_{14}x^2 + 2a_{24}x + a_{34},$$

$$y_4'' = 12a_{04}x^2 + 6a_{14}x + 2a_{24},$$

$$y_4''' = 24a_{04}x + 6a_{14},$$

$$y_4^{IV} = 24a_{04},$$

so that

$$24a_{04} - 4x(24a_{04}x + 6a_{14}) + (4x^2-4)(12a_{04}x^2 + 6a_{14}x + 2a_{24})$$

$$+ 4x(4a_{04}x^3 + 3a_{14}x^2 + 2a_{24}x + a_{34})$$

$$-64(a_{04}x^4 + a_{14}x^3 + a_{24}x^2 + a_{34}x + a_{44}) = 0,$$

$$0 \cdot a_{04}x^4 - 28a_{14}x^3 - (144a_{04} + 48a_{24})x^2$$

$$-(48a_{14} + 60a_{34})x + (24a_{04} - 8a_{24} - 64a_{44}) = 0.$$

Thus, a_{04} is arbitrary, $a_{14} = 0$, $a_{24} = -3a_{04}$, $a_{34} = 0$ and $a_{44} = (3/4)a_{04}$. Consequently,

$$y_4 = a_{04}(x^4 - 3x^2 + \tfrac{3}{4}).$$

Further solutions y_n, $n = 5,6,7,\ldots$, can be determined in a similar manner.

The preceding solution set forms an orthogonal system, with respect to the weight function e^{-x^2}, over the interval $(-\infty,\infty)$. This set of polynomials is thus considered as an analogue to a set of Hermite polynomials.

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BIOGRAPHICAL SKETCH

John Wesley Meux was born April 25, 1928, at Little Rock, Arkansas. In June, 1946, he was graduated from Prescott High School in Prescott, Arkansas. He enlisted in the United States Marine Corps in August, 1946, and served until August, 1949. In May, 1951, he was commissioned in the United States Army Reserve and is still a member of the active Reserve. He received the degree of Bachelor of Science in June, 1953, from Henderson State Teachers College. During the following year he was employed by Northwestern National Life Insurance Company, which position he left to teach mathematics at Little Rock Central High School, Little Rock, Arkansas. In June, 1955, he enrolled in the Graduate School of the University of Arkansas and received the degree of Master of Science from that university in June, 1957. He was employed as a graduate assistant in the Department of Mathematics during this period. In June, 1957, he entered the Graduate School of the University of Florida and worked as a graduate assistant in the Department of Mathematics until June, 1959. Since that time he has been employed as an instructor in the Department of Mathematics while pursuing his work toward the degree of Doctor of Philosophy.

John Wesley Meux is married to the former Elizabeth Whitten and is the father of two children, Jay and Laurie. He is a member of the Mathematical Association of America.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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